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Correlation Dimension

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In many situations, both deterministic and probabilistic, one can develop further the study of the multifractal structure of a dynamical system, particularly when there exist strange attractors. Multifractal refers to a notion of size emphasizing the variations of the weigth of the measure. In such schemes, one has to compute a free energy function associated to some sequence of partitions. We relate the free energy function, associated to a sequence of uniform partitions of exponentially decreasing diameters, and the correlation dimension which refers to a quantity that is the most accessible in numerical computations. Finally we discuss of two assumptions for the existence of free energy functions.

KEY WORDS: Correlation dimension; multifractal; thermodynamic formalism; free energy function.

INTRODUCTION

Let (X, μ, g) be a dynamical system where X is a metric compact space, g a transformation $X \rightarrow X$ and μ a g-invariant ergodic measure on X. We define now a natural quantity

$$C(r, 1) = \frac{\operatorname{Log} \int \mu(B(x, r))\, \mu(dx)}{\operatorname{Log} r} \tag{0.1}$$

which is the most accessible in numerical computations based on time-series of a dynamical system,[GHP, P] and we are interested, when it exists, in the value

$$C = \lim_{r \rightarrow 0} C(r, 1)$$

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We extend these notions to a real valued function for any real β by

$$C(r, \beta) = \frac{\text{Log} \int \mu(B(x, r))^{\beta} \mu(dx)}{\text{Log}\, r} \tag{0.2}$$

and when it exists

$$\forall \beta \in \mathbb{R},\ C(\beta) = \lim_{r \to 0} C(r, \beta)$$

This function, known as the correlation dimension[GHP, HP, O, P, PT, Si2, Si3] arises in the numerical investigation of strange attractors and other models involving fractal measures for $\beta = 1$, and differs in general with other characteristic dimensions.

Let us follow the approach suggested by D. Ruelle and described in ref. [P]. Let then the space $Y = X \times X$ equipped with the metric

$$d_1((x, y), (z, t)) = d(x, z) + d(y, t)$$

—d is the metric on X—and the direct product measure

$$\nu = \mu \times \mu$$

Let then the diagonal

$$D = \{(x, x) \in Y\} \qquad \text{and} \qquad D_r = \{y \in Y / d_1(y, D) \leqslant r\} \qquad \text{for} \quad r > 0$$

We have then

$$\nu(D_r) = \int_X \mu(B(x, r))\, \mu(dx)$$

and therefore

$$\frac{\text{Log}\, \nu(D_r)}{\text{Log}\, r} = \frac{\text{Log} \int \mu(B(x, r))\, \mu(dx)}{\text{Log}\, r} = C(r, 1) \tag{0.3}$$

Up to now, no dynamics was involved. A dynamical interpretation is the following. Let g be continuous and preserving a Borel normalized ergodic measure. Given $(x, y) \in X \times X$, let

$$C(x, y, n, r) = \frac{2}{n(n-1)} \text{Card}\{(i, j)/d(g^i(x), g^j(y)) \leqslant r \\ \text{for} \quad 0 \leqslant i < j < n\}$$

The measure ν is invariant and ergodic for the $\mathbb{Z}^2$-action $g_{i,j}(x, y) = (g^i(x), g^j(y))$. It is proved in ref. [P] that for ν-a.e. point (x, y)

$$\lim_{n \to +\infty} C(x, y, n, r) = \nu(D_r) \sim \begin{cases} r^{C^+} \\ r^C \\ r^{C^-} \end{cases}$$

which means that

$$C^+ = \overline{\lim_{r \to 0}} \frac{\operatorname{Log} C(x, y, n, r)}{\operatorname{Log} r} \quad \text{and} \quad C^- = \underline{\lim_{r \to 0}} \frac{\operatorname{Log} C(x, y, n, r)}{\operatorname{Log} r}$$

and when the limit exists $C = C^+ = C^-$ following (0.1) and (0.3).

We know that in many examples, included all the ones we study, we have

$$\lim_{r \to 0} \frac{\operatorname{Log} \mu(B(x, r))}{\operatorname{Log} r} = \alpha'\mu \text{ a.s.} \left(= \frac{h_\mu}{\chi_\mu} \text{ in dimension } 1 \right) \tag{0.4}$$

refs. [CLP, HJKPS, M, Ma, O, R, Si1, Y], (where h_μ denotes the entropy and χ_μ the Lyapunov exponent).

In many examples studied intensively,[BMP, CLP, EM, HJKPS, HW, Si1] the value α' in (0.4) satisfies $\alpha' = F'(1)$ where F is a concave free energy function, defined for any real β (when it exists) from a sequence of partition functions $(Z_n)_{n \geqslant 1}$

$$F(\beta) = -\lim_{n \to +\infty} \frac{1}{n} \operatorname{Log}_m Z_n(\beta) \tag{0.5}$$

where

$$Z_n(\beta) = \sum_{\substack{U \in U_n \\ \mu(U) > 0}} \mu(U)^\beta \tag{0.6}$$

is derived from a sequence of partitions $(U_n)_{n \geqslant 1}$ whose diameters tend to 0 when $n \to +\infty$ ($\# U_n = m^n$). By a simple argument (when F' exists) we get

$$\exists \theta \in]0; 1[, F(2) = F(1) + F'(1 + \theta) = F'(1 + \theta)$$

We have also[CLP, R, Si1]

$$F(2) \ (= F'(1 + \theta)) \geqslant \alpha' \ (= F'(1)) \tag{0.7}$$

(= in the degenerate case when F is linear, and $>$ in the general case when F is strictly concave) and it is proved[Si2, Si3] in general situations that $F(2) = C(1)$.

Our aim is to prove the variational principle (Theorems 2.1 and 2.5):

$$\forall \beta \in \mathbb{R},\ C(\beta) = F(\beta + 1) \tag{0.8}$$

in some significant cases when the partitions used are uniform (when F exists!).

For one-dimensional expanding Markov maps (see 1.2), we have proved the relation (0.8) in ref. [Si3]. In the same paper, it was easily generalized to the case of Axiom A surface diffeomorphisms.

We found in ref. [Si2] a very easy proof of this statement for a two-dimensional dynamical system: the Sierpinski carpet (see 1.5),[O] which is very suitable for the calculations.

In the previous cases we were able to compute the two functions and therefore to compare them. In fact, by adapting the proof, we realize that the statement was more general.

Nevertheless in all the cases, the most important step, often proved by the hard way with large deviations theorems, is the existence of the free energy function F (there is also a dynamical free energy, the pressure, associated to the dynamical partition: the so-called Markov partition, more intrinsic, easier to compute and more regular). This existence is discussed in the Section 4 where we give two assumptions which are satisfied by many relevant cases in the literature of different domains in sciences.

1. EXAMPLES

For all dynamical systems presented here there exists a free energy function, given explicitely in some cases, although the partitions which appear naturally (dynamically) are not generally uniform. The Example 1.1 is used in Section 3.

1.1. Multiplicative Chaos

One studies a class of random measures obtained by random iterated multiplications. To this model corresponds a rigorous study of the phase transition of a system with random iteractions. This occurs in different domains of physics and chemistry:[CK, CD, DS, F, HW, K, KP] polymers, spin glasses, thermodynamic, turbulence, traveling waves, rainfall distribution...

Consider the interval $I = [0; 1]$, an integer $b \geqslant 2$, W a non negative with mean 1 random variable, and $\{W(i_1, i_2, ..., i_n)/n \in \mathbb{N}^*,\ \forall k \in \mathbb{N}^*,$

$0 \leqslant i_k < b\}$ a sequence of i.i.d. random variables distributed like W. Let μ_n be the measure defined on $[0; 1[$ (on the uniform partitions) in the following way:

$$\varphi_n(x) = W(i_1)\, W(i_1, i_2) \cdots W(i_1, i_2, ..., i_n)$$
$$\text{for} \quad x \in I(i_1, i_2, ..., i_n) = \left[\sum_{k=1}^{n} i_k b^{-k}; \sum_{k=1}^{n} i_k b^{-k} + b^{-n} \right[$$

then μ_n is defined by its density

$$\mu_n(dx) = \varphi_n(x)\, dx \tag{1.1.1}$$

The representation theorem of martingales (with respect to the proper σ-algebra $\mathscr{F}_n = \sigma\{ W(i_1, i_2, ..., i_n)/0 \leqslant i_k < b\}$) assures us the existence of the limit

$$\mu = \lim_{n \to +\infty} \mu_n \tag{1.1.2}$$

We define the measure μ_n^β associated to the random variable W^β like μ_n was associated to W, and let

$$Z_n(\beta) = -\frac{1}{n} \operatorname{Log}\{\mu_n^\beta([0; 1[)\}$$

We can present this model (isomorphically) with the b-tree construction (the multiplicative cascade): the indexes $(i_1, i_2, ..., i_n)$ code a branch of length n, and the random variable $W = e^Y$. We take the sum over all the branches t of length n, and $S_n(t)$ the sum of the i.i.d. random variables Y_{i_j} of the vertices met along the branch t, and this gives

$$Z_n(\beta) = -\frac{1}{n} \operatorname{Log} \sum_{|t|=n} e^{\beta S_n(t)} \tag{1.1.3}$$

If we take Y upper and lower bounded, then there exists a limit in (1.1.3) when n goes to $+\infty$,[HW, KP] and the limit is a free energy function F associated to a sequence of uniform partitions $(U_n)_{n \geqslant 1}$ of diameters b^{-n}.

1.2. Expanding Markov Maps

Let X be a closed interval, $[0; 1]$, or the circle S^1, and g a Markov map on X.[CLP] This example was probably the first studied rigorously from the point of view of multifractal properties. Particularly it has been proved the existence of a C^2 free energy function, and by reducing the problem from dimension 2 to dimension 1 it is in fact an analytic free energy function associated to a sequence of uniform partitions.[Si1]

1.3. Cookie-cutters

Let $X=[0;1]$ and I_0 and I_1 two disjoint closed subintervals of X. A cookie-cutter[R] is a $C^{1+\alpha}$ map $g\colon I_0 \cup I_1 \to \mathbb{R}$ such that

$$|g'|>1 \qquad \text{and} \qquad g(I_0)=g(I_1)=X$$

Moreover, the measure μ is the Gibbs measure of a real Hölder continuous function $\phi\colon X\to\mathbb{R}$. Up to the regularity this is a subcase of 1.2 and the results are similar.

1.4. Axiom A Diffeomorphisms

We study here a two-dimensional dynamical system.[B, Ru, Si1] Let X be a compact manifold of dimension 2 (for example the torus) and g a C^2 axiom A diffeomorphism. The g-invariant measure μ is the Bowen–Margulis measure—the one that realizes the maximum of topological entropy—or the Gibbs measure of a real Hölder continuous function $\phi\colon \Lambda\to\mathbb{R}$ (Λ basic set).

One finds in ref. [Si1] explicit formulas of the free energy function associated to a sequence of uniform partitions and the two quantities in (0.8) are shown to be equal.

1.5. The Sierpinski Carpet

This is an example of a two-dimensional dynamical system contracting in the two directions with different ratios. The Sierpinski carpets are planar generalized Cantor sets and are defined on $[0;1]^2$.[Mc, O, Si2] Given integers $n\geqslant m$ and a set

$$S\subset\{(i,j)/0\leqslant i<n \text{ and } 0\leqslant j<m\}$$

with $\#(S)=c$, we define the fractal set

$$\bar{S}=\left\{\left(\sum_{k\geqslant 1}\frac{x_k}{n^k},\sum_{k\geqslant 1}\frac{y_k}{m^k}\right)\Big/\forall k,(x_k,y_k)\in S\right\} \tag{1.5.1}$$

The measures μ that are considered are Gibbs measures. It has been calculated explicitely in ref. [Si2] the two functions, and therefore we proved (0.8) very easily.

1.6. Recursive Digraph Fractals

One starts from a directed multigraph (V, E).[EM] The set E is composed of the edges of the graph, and the elements of V are the vertices. This graph is supposed to be strongly connected, that is, there is a path from any vertex to any other along the edges of the graph (there are also two edges leaving each node). A digraph recursive fractal is based on seed sets, nonempty compact subsets of $\mathbb{R}^n$ with an usual "open set condition," J_u for each $u \in V$, and ratios $r(e) \in]0; 1[$ (which represent homotheties in $\mathbb{R}^n$). The measure μ that we use is of Markov type and is defined recursively. We can prove for this n-dimensional dynamical system the existence of a spectral radius[EM] which is $-F$.

2. VARIATIONAL PRINCIPLE

We suppose that our dynamical system satisfies the existence of a free energy function F (0.5) associated to a uniform partition $(U_n)_{n \geqslant 1}$ of diameters b^{-n} where $b > 1$. We have then

Theorem 2.1. We have

$$C(1) = \overline{\lim_{r \to 0}}\, C(r, 1) = F(2)$$

We consider for convenience the quantities $L(r) = -C(r, 1)$ and we have obviously for any real $a > 1$

Proposition 2.2. The sequence $(L(a^{-n}))_{n \geqslant 1}$ is convergent if and only if

$$\lim_{r \to 0} L(r) \qquad \text{exists}$$

Then Theorem 2.1 follows directly from Lemmas 2.3 and 2.4. We give here a proof in the case where we study a one-dimensional dynamical system, and we easily generalize in the remark. The first step is to compute an upper bound of the supremum limit which is given by

Lemma 2.3. We have

$$\overline{\lim_{r \to 0}}\, L(r) \leqslant -F(2)$$

Proof. Let us compute for any integer n

$$L(b^{-n}) = \frac{1}{n} \operatorname{Log}_b \int \mu(B(x, b^{-n}))\, \mu(dx)$$

$$= \frac{1}{n} \operatorname{Log}_b \left\{ \sum_{U \in U_n} \int_U \mu(B(x, b^{-n}))\, \mu(dx) \right\} \tag{2.1}$$

For any $U \in U_n$ (with at most two exceptions), we take respectively the left neighbour V and the right neighbour W in the partition. We have obviously for any $x \in U$, $B(x, b^{-n}) \subset V \cup U \cup W$, and we get

$$\int_U \mu(B(x, b^{-n}))\, \mu(dx) \leqslant (\mu(V) + \mu(U) + \mu(W))\, \mu(U)$$

$$\leqslant 3(\mu(V)^2 + \mu(U)^2 + \mu(W)^2) \tag{2.2}$$

Comparing (2.1) and (2.2), we obtain therefore

$$L(b^{-n}) \leqslant \frac{1}{n} \operatorname{Log}_b 9 \sum_{U \in U_n} \mu(U)^2 \tag{2.3}$$

Taking the upper limit in (2.3), we obtain the result since the right-hand side converges to $-F(2)$. ▌

Remark. In higher dimension $m \in \mathbb{N}^*$, the number of neighbours V (included U) which appear in (2.2) is 3^m, and the constant which appears in (2.3) is 9^m. Hence we get the same result.

We establish a reciprocal since we obtain a lower bound of the infimum limit in

Lemma 2.4. We have

$$\varliminf_{r \to 0} L(r) \geqslant -F(2)$$

Proof. We compute like in (2.1) for any integer n

$$L(b^{-n}) = \frac{1}{n} \operatorname{Log}_b \left\{ \sum_{U \in U_n} \int_U \mu(B(x, b^{-n}))\, \mu(dx) \right\}$$

Since we have for any $U \in U_n$ $|U| = b^{-n}$, we get for any $x \in U$

$$U \subset B(x, b^{-n}) \qquad \text{and} \qquad \mu(U) \leqslant \mu(B(x, b^{-n}))$$

hence we obtain finally

$$L(b^{-n}) \geqslant \frac{1}{n} \operatorname{Log}_b \sum_{U \in U_n} \mu(U)^2 \tag{2.4}$$

Taking the lower limit in (2.4), we obtain the result since the right-hand side converges to $-F(2)$. ▮

Remark. In higher dimension $m \in \mathbb{N}^*$, we have not $\forall x \in U$, $U \subset B(x, b^{-n})$. It suffices to replace b^{-n} by $(b-\varepsilon)^{-n}$ for $\varepsilon > 0$, and then for any integer $n \geqslant N(m, \varepsilon)$

$$\forall U \in U_N, \forall x \in U, U \subset B(x, (b-\varepsilon)^{-n})$$

and we obtain the same inequality when ε goes to 0.

Mixing the results of Lemmas 2.3 and 2.4, we get the existence of $C(1)$

$$C(1) = -\lim_{r \to 0} L(r) = F(2)$$

We can easily generalize with

Theorem 2.5. We have for any real β

$$C(\beta) = \lim_{r \to 0} C(r, \beta) = F(\beta + 1)$$

For $\beta = 0$, both quantities $C(0)$ and $F(1)$ are 0. It is clear that the result follows for any positive β and that for $\beta < 0$, the proofs are also analogous. ▮

3. LOCAL DIMENSION

The dimension of a measure is directly related to entropy and Lyapunov exponents.[Ma, Y] The unstable directions are crucial since there is no contribution in neutral or contracting directions for entropy.[LY1] Nevertheless, neutral and contracting directions have influence on dimension.[LY2]

One can ask whether or not there exist assumptions for the existence of the almost sure value of the local dimension (0.4). For example, if the measure μ is a finite Borel measure on $\mathbb{R}$ (or $\mathbb{R}^n$), one proves, via the Besicovitch covering lemma[G, LY1] that for μ-a.e. x

$$0 < \inf_{0 < r \leqslant 1} \frac{\mu(B(x, r))}{r} \left(\inf_{0 < r \leqslant 1} \frac{\mu(B(x, r))}{r^n} \right)$$

and

$$\overline{\lim_{r \to 0}} \frac{\operatorname{Log} \mu(B(x, r))}{\operatorname{Log} r} \leqslant 1 \ (\leqslant n)$$

There is a conjecture of Eckmann and Ruelle[ER] claiming that if g is a $C^{1+\alpha}$ diffeomorphism with non-zero Lyapunov exponents,[Si1, Y] then (0.4) holds for μ-a.e. $x \in X$. On the other hand Ledrappier and Misiurewicz[LM] constructed an example of a smooth map on $[0; 1]$ preserving an ergodic measure for which (0.4) does not hold on a set of positive Lebesgue measure (see also Cutler[C] where the local dimension depends on x or ref. [S]).

Good schemes seem to be hyperbolic dynamical systems; in dimension 1, 1.1, 1.2, 1.3 and ref. [L]: with Gibbs measures;[CLP, HJKPS, HW, R] in dimension 2, 1.4 and 1.5: with Gibbs measures or Markovian measures[M, O, Si1] or with invariant ergodic measures with non-zero Lyapunov exponents;[Y] in dimension n, 1.6: with Markovian measures.

We can notice that it is possible to construct examples where the differences between all dimensions are very sharp.[Y]

We see, with some counter examples, that it is not easy to get general results to the following questions.

Question 1. Does the existence of the almost sure local dimension (0.4) α' imply the existence of the free energy function F? The answer is no.

We take the dynamical system defined in 1.1, the multiplicative chaos, using the same notations. With the definition of the measure μ, we can compute α' since

$$\begin{aligned}
\frac{\operatorname{Log} \mu[I(i_1, i_2, \ldots, i_n)]}{\operatorname{Log} b^{-n}} &= -\frac{1}{n} \operatorname{Log}_b W(i_1)\, W(i_1, i_2) \cdots W(i_1, i_2, \ldots, i_n) \\
&= -\frac{1}{n} \sum_{k=1}^{n} Y(i_1, i_2, \ldots, i_k) \\
&= -\frac{1}{n} \sum_{k=1}^{n} Y_k \xrightarrow[n \to +\infty]{} -\mathbb{E}(Y) = \alpha'
\end{aligned}$$

where we have $W(i_1, i_2, \ldots, i_k) = b^{Y(i_1, i_2, \ldots, i_k)}$ and the $(Y_k)_{k \geqslant 1}$ are i.i.d. random variables satisfying $1 = \mathbb{E}(W) = \mathbb{E}(e^Y)$ $(\mathbb{E}(Y) \leqslant 0)$.

Following refs. [F, HW, KP], let for any real β

$$\chi_b(\beta) = \operatorname{Log} \mathbb{E}(W^\beta) - (\beta - 1)$$

where W is taken with exponential moments. Then there exists a free energy function F and a critical value in $\mathbb{R}$ (temperature since $\beta = 1/kT$)

$$\beta_c = \sup\{\beta \geqslant 1/\chi_b(\beta) < 0\}$$

such that F is C^∞ everywhere except at β_c where it is not even C^2 ($\beta_c < +\infty \Leftrightarrow b\mathbb{P}(Y = \text{esssup } Y) < 1$ where $\text{esssup } Y = \sup\{y \in \mathbb{R}/\mathbb{P}(Y < y) < 1\}$). If we have some restrictions on the exponential moments of the random variable W, then the function F will not be defined on a part of $\mathbb{R}$.

Question 2. Does the existence of the free energy function imply the existence of an almost surely unique local dimension? The answer is no.

Let the metric space $X = [0; 1]^2$ and the measure $\mu = 1/2\lambda_1 \times \delta_0 + 1/2\lambda_2$ where λ_1 is one-dimensional Lebesgue measure on $[0; 1]$, δ_0 is the Dirac unit mass measure at 0 and λ_2 is two-dimensional Lebesgue measure on X.[HW]

We compute the free energy function F and we obtain

$$\forall \beta \in \mathbb{R}, F(\beta) = (\beta - 1)\, 1_{]-\infty; 1]}(\beta) + 2(\beta - 1)\, 1_{[1; +\infty[}(\beta)$$

We show that on two sets of μ-measure 1/2, the limits in (0.4) are respectively 1 and 2 (which are respectively the left derivative of F at 1 and the right derivative of F at 1 - compare with (0.7)). Actually it is easy to verify that for a point $M(x, y)$ of X we have:

- If $M(x, y)$ satifies $y = 0$ then we get

$$\lim_{r \to 0} \frac{\operatorname{Log} \mu(B(M, r))}{\operatorname{Log} r} = 1 \qquad \text{and} \qquad \mu(\{M(x, 0) \in [0; 1]^2\}) = \frac{1}{2}$$

- If $M(x, y)$ satifies $y > 0$ then we get

$$\lim_{r \to 0} \frac{\operatorname{Log} \mu(B(M, r))}{\operatorname{Log} r} = 2 \qquad \text{and} \qquad \mu(\{M(x, y) \in [0; 1]^2/y > 0\}) = \frac{1}{2}$$

With some regularity of the free energy function F, for example at least C^2, have we $\alpha' = F'(1)$?

4. EXISTENCE OF FREE ENERGIES

A good scheme in the study of free energy functions is an ergodic dynamical system (X, μ, T) where X is a compact metric space, T is a map onto X ($C^{1+\delta}$, C^2, hyperbolic, expanding...) and μ is a Gibbs measure—a T-invariant Borel probability measure—associated to a potential which is a δ-Hölder continuous function ϕ. Such dynamical systems are very common and relevant for the study of physical systems. We shall use a constructive method based on large deviations on the Markov partitions $(Q_n)_{n \geqslant 1}$ which is given by the iterations of the dynamical partition.

We shall discuss of two assumptions.

Assumption 1. There exist two constants a and b (<0) and an integer $N \geqslant 1$ such that for any integer $n \geqslant N$ and any $U \in Q_n$ with $\mu(U) \neq 0$

$$a \leqslant \frac{1}{n} \operatorname{Log} \mu(U) \leqslant b$$

Remark. This assumption if satisfied by a large class of measures, namely Gibbs measures. Let $\phi \in C^{\delta}(X)$ the potential associated to the measure μ. We get uniformly in n

$$\inf_X \phi \leqslant \frac{1}{n} \operatorname{Log} \mu(U) \leqslant \sup_X \phi < 0$$

Assumption 2. There exist two constants c and d (<0) and an integer $N \geqslant 1$ such that for any integer $n \geqslant N$ and any $U \in Q_n$

$$c \leqslant \frac{1}{n} \operatorname{Log} |U| \leqslant d$$

Remark. This assumption is satisfied by a large class of maps (hyperbolic: $C^{1+\delta}$ expanding Markov maps, axiom A diffeomorphisms, conformal expanding maps, recursive digraph fractals...) since the dilatation of the elements of the dynamical partition is controlled by $\varepsilon \operatorname{Log} T'$ $(<0$ for $\varepsilon = \pm 1)$ which is supposed Hölder continuous on the compact X. We get uniformly in n

$$\inf_X \varepsilon \operatorname{Log} T' \leqslant \frac{1}{n} \operatorname{Log} |U| \leqslant \sup_X \varepsilon \operatorname{Log} T' < 0$$

We try to show that these assumptions are relevant. The idea is to construct, at any rank n and for any real β, sets of elements of Q_n where the distribution of the mass of the $\mu(U)^\beta$ of the partition functions (0.6) is the larger (in the scale $1/n$ Log), and is the place where the large deviations occur.

For the Assumption 1, let us define for any integer $n \geqslant 1$ the sequences

$$a_n = \inf_{U \in Q_n} \frac{1}{n} \operatorname{Log} \mu(U) \qquad \text{and} \qquad b_n = \sup_{U \in Q_n} \frac{1}{n} \operatorname{Log} \mu(U),$$

and at the limit

$$a = \varliminf_{n \to +\infty} a_n \qquad \text{and} \qquad b = \varlimsup_{n \to +\infty} b_n$$

1. assume $a = -\infty$: there exist a subsequence $(n_k)_{k \geqslant 1}$ and elements $A_k \in Q_{n_k}$ such that $a_{n_k}(-k) \to -\infty$ and $\mu(A_k) \leqslant \exp -kn_k$. Take any real $\beta < 0$ and compute the following

$$\begin{aligned} \frac{1}{n_k} \operatorname{Log} Z_{n_k}(\beta) &= \frac{1}{n_k} \operatorname{Log} \sum_{U \in Q_{n_k}} \mu(U)^\beta \\ &\geqslant \frac{1}{n_k} \operatorname{Log} \mu(A_k)^\beta \geqslant \beta a_{n_k} \xrightarrow[k \to +\infty]{} +\infty \end{aligned}$$

This means that the free energy function is degenerate $(+\infty)$ for $\beta < 0$, and for interesting cases a is finite. This gives the left-hand side.

2. assume $b = 0$: there exist a subsequence $(n_k)_{k \geqslant 1}$ and elements $B_k \in Q_{n_k}$ such that $(-2/k) < b_{n_k} \leqslant 0$ and $(-1/k) < (1/n_k) \operatorname{Log} \mu(B_k) \leqslant 0$. We get for $\beta \geqslant 1$ (since we have the following inequality $\mu(U)^\beta \leqslant \mu(U)$)

$$\mu(B_{n_k})^\beta \leqslant Z_{n_k}(\beta) \leqslant Z_{n_k}(1) = 1$$

and then

$$-\frac{1}{k} \leqslant \frac{1}{n_k} \operatorname{Log} Z_{n_k}(\beta) \leqslant 0$$

which means that the free energy function is degenerate $(=0)$ for $\beta \geqslant 1$. Therefore we take $b < 0$ and this gives the right-hand side.

These two cases lead us to consider Assumption 1. The Assumption 2 is treated similarly.

In the following we try to motivate these assumptions in quite general situations. The idea is to construct sets included in the partitions Q_n which are made of elements of "same" measure and "same" length which are preponderous in the computation of the partition functions and by the way to the free energy function.

By Assumption 1 we can define for any integer $i \in [an; bn-1]$ $(i<0)$ and for any real β the sets (since elements of Q_n satisfy: $(1/n)\operatorname{Log}\mu(U) \in [a; b]$) which are empty for integer $i \notin [an; bn-1]$

$$E_{n(i)} = \{U \in Q_n / \operatorname{Log}\mu(U) \in [i; i+1[\,\}$$

The elements of this set have "same" measure since we have $e^i \leqslant \mu(U) \leqslant e^{i+1}$. From (0.6), the partitions functions satisfy for any real β

$$Z_n(\beta) = \sum_{U \in Q_n} \mu(U)^\beta = \sum_i \sum_{U \in E_{n(i)}} \mu(U)^\beta$$

Define the integer $n(i, \beta)$ such that for any $\beta \in \mathbb{R}$

$$\sum_{U \in E_{n(i)}} \mu(U)^\beta \leqslant \sum_{U \in E_{n(i,\beta)}} \mu(U)^\beta$$

and we have obviously since the integer i varies in a linear scale

$$\sum_{U \in E_{n(i,\beta)}} \mu(U)^\beta \leqslant Z_n(\beta) = \sum_i \sum_{U \in E_{n(i)}} \mu(U)^\beta \leqslant (b-a)\, n \sum_{U \in E_{n(i,\beta)}} \mu(U)^\beta$$

and this gives

$$\begin{aligned}\frac{1}{n}\operatorname{Log} \sum_{U \in E_{n(i,\beta)}} \mu(U)^\beta &\leqslant \frac{1}{n}\operatorname{Log} Z_n(\beta)\\ &\leqslant \frac{1}{n}\operatorname{Log}[(b-a)n] + \frac{1}{n}\operatorname{Log} \sum_{U \in E_{n(i,\beta)}} \mu(U)^\beta\end{aligned}$$

Therefore we obtain

$$\frac{1}{n}\operatorname{Log} Z_n(\beta) = \frac{1}{n}\operatorname{Log} \sum_{U \in Q_n} \mu(U)^\beta \sim \frac{1}{n}\operatorname{Log} \sum_{U \in E_{n(i,\beta)}} \mu(U)^\beta$$

where we have the following

$$(e^{n(i,\beta)})^\beta \,\#E_{n(i,\beta)} \leqslant \sum_{U \in E_{n(i,\beta)}} \mu(U)^\beta \leqslant (e^{n(i,\beta)+1})^\beta \,\#E_{n(i,\beta)}$$

We get finally

$$\frac{1}{n}\operatorname{Log} Z_n(\beta) \sim \frac{1}{n}\operatorname{Log} \# E_{n(i,\beta)} + \beta\,\frac{n(i,\beta)}{n} \tag{4.1}$$

where the set $E_{n(i,\beta)}$ satisfies the desire property.

Consider the Borel probability measures

$$\gamma_n(\beta) = \frac{1}{\# E_{n(i,\beta)}} \sum_{U \in E_{n(i,\beta)}} \frac{1}{|U|} \mathbb{1}_U$$

and

$$\eta_n(\beta) = \frac{1}{n(i,\beta)} \sum_{j=0}^{-n(i,\beta)+1} T^j \gamma_n(\beta)$$

which indicate the distribution of the set $E_{n(i,\beta)}$. Since the set $M(X)$ is a compact set there exists a subsequence $(n_k)_{k \geqslant 1}$ such that

$$\gamma_{n_k}(\beta) \xrightarrow[k \to +\infty]{} \gamma_\beta \in M(X) \qquad \text{and} \qquad \eta_{n_k}(\beta) \xrightarrow[k \to +\infty]{} \eta_\beta \in M_T(X)$$

We introduce these measures since they indicate where the distribution of the mass of the $\mu(U)^\beta$ of the partitions is the larger. Notice that we consider the length of the elements U. To this purpose we want to control the distribution of the length and this is where we use assumption 2. The idea is to choose preponderous elements of "same" length among the elements of $E_{n(i,\beta)}$, and this is done by the same fashion.

Let us define for any integer $j \in [cn; dn-1]$ and for any real β the sets

$$H_{n(j)} = \{ U \in E_{n(i,\beta)} / \operatorname{Log} |U| \in [j; j+1[\}$$

The elements of this set have "same" measure since they belong to the set $E_{n(i,\beta)}$. Moreover they have "same" length since we have: $e^j \leqslant |U| \leqslant e^{j+1}$.

Let $n(j,\beta)$ the integer among the $n(j)$ such that for any $\beta \in \mathbb{R}$

$$\sum_{U \in H_{n(j)}} \mu(U)^\beta \leqslant \sum_{U \in H_{n(j,\beta)}} \mu(U)^\beta$$

We have then

$$\sum_{U \in H_{n(j,\beta)}} \mu(U)^\beta \leqslant \sum_j \sum_{U \in H_{n(j)}} \mu(U)^\beta = \sum_{U \in E_{n(i,\beta)}} \mu(U)^\beta \leqslant (d-c)\,n \sum_{U \in H_{n(j,\beta)}} \mu(U)^\beta$$

and therefore with (4.1) (since $1/n \operatorname{Log}(d-c)n$ tends to 0 at the limit)

$$\frac{1}{n}\operatorname{Log} Z_n(\beta) \sim \frac{1}{n}\operatorname{Log}\sum_j \sum_{U \in H_{n(j)}} \mu(U)^\beta \sim \frac{1}{n}\operatorname{Log}\sum_{U \in H_{n(j,\beta)}} \mu(U)^\beta$$

which implies finally

$$\frac{1}{n}\operatorname{Log} Z_n(\beta) \sim \frac{1}{n}\operatorname{Log}\ \# H_{n(j,\beta)} + \beta\frac{n(i,\beta)}{n} \tag{4.2}$$

since we have by definition

$$(e^{n(i,\beta)})^\beta\ \# H_{n(j,\beta)} \leqslant \sum_{U \in H_{n(j,\beta)}} \mu(U)^\beta \leqslant (e^{n(i,\beta)+1})^\beta\ \# H_{n(j,\beta)}$$

Remark that the different sets $H_{n(j,\beta)}$ are composed of elements of "same" measure $\exp\{n(i,\beta)\}$ and "same" length $\exp\{n(j,\beta)\} \sim b(n,\beta)^{-n}$ (in the scale $1/n$ Log), and it is where the large deviations occur. This leads to

$$\frac{1}{n}\operatorname{Log}_{b(n,\beta)} Z_n(\beta) \sim \frac{1}{n(j,\beta)}\operatorname{Log}\ \# H_{n(j,\beta)} + \beta\frac{n(i,\beta)}{n(j,\beta)} \tag{4.3}$$

Consider the Borel probability measures

$$\omega_n(\beta) = \frac{1}{\# H_{n(j,\beta)}} \sum_{U \in H_{n(j,\beta)}} \frac{1}{|U|} \mathbb{1}_U$$

and

$$\zeta_n(\beta) = \frac{1}{n(j,\beta)} \sum_{j=0}^{-n(j,\beta)+1} T^j \omega_n(\beta)$$

Since $M(X)$ is a compact set there exists a subsequence $(n_k)_{k \geqslant 1}$ such that

$$\omega_{n_k}(\beta) \xrightarrow[k \to +\infty]{} \omega_\beta \in M(X) \qquad \text{and} \qquad \zeta_{n_k}(\beta) \xrightarrow[k \to +\infty]{} \zeta_\beta \in M_T(X)$$

Recall that ζ_β is the distribution of uniform elements $U \in H_{n(j,\beta)}$ which cover the subset of X which is preponderous for the distribution of the mass $\mu(U)^\beta$. The sequences in (4.3) $(1/n)\operatorname{Log}\# H_{n(j,\beta)}$, $(n(i,\beta)/n)$ and

$(n(j, \beta)/n)$ have connections with the measure ζ_β. The basis b in the following expression depends on β

$$F(\beta) = \lim_{n \to +\infty} -\frac{1}{n} \operatorname{Log}_b Z_n(\beta)$$

and is given by the limit of the sequence $-(n(j, \beta)/n) = b(n, \beta)$.

We can give explicit expressions of the previous statements in many situations. For example in the case where μ is the Gibbs measure of the potential $\phi \in C^\delta(X)$ and the map T is hyperbolic (for example one-dimensional expanding Markov maps [CLP]) and such that $-\operatorname{Log} T \in C^\delta(X)$, we get for any real β

$$\frac{n(i, \beta)}{n} \xrightarrow[n \to +\infty]{} \int_X \phi \, d\xi_\beta \qquad \text{and} \qquad \frac{n(j, \beta)}{n} \xrightarrow[n \to +\infty]{} \int_X \varepsilon \operatorname{Log} T' \, d\xi_\beta$$

and $F(\beta)$ is connected with thermodynamic quantities[Si1] involving entropy, pressure and Lyapunov exponents.

Unfortunately since the free energy function F is a large deviations functional, one needs some regularity in the sens that it has to satisfy some smooth properties. All the examples that we can study are too rigid, the assumptions are too strong so that they generate too much regularity for the models presented.

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